

Analysis of the turbulent law of the wall through the finite scale Lyapunov theory

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Abstract

This work analyzes the turbulent velocity distribution in proximity of the wall using the finite-scale Lyapunov theory just presented in previous works. This theory is here applied to the steady boundary layer under the hypothesis of moderate pressure gradient and fully developed flow along the streamwise direction. The analysis gives an equation for the velocities correlation and identifies the parameters of the expression of the average velocity through the statistical properties of the velocity correlation functions. In particular, the von Kármán constant, theoretically calculated, is about 0.4, and the dimensionless Prandtl's length is in function of the Taylor-scale Reynolds number. The study provides the average velocity distribution and gives also the variation laws of the other variables, such as Taylor scale and Reynolds stress. The obtained results show that the finite-scale Lyapunov theory is adequate for studying the turbulence in the proximity of the wall.

Keywords:

Finite-scale Lyapunov theory, von Kármán constant, Law of the wall.

1. Introduction

The average velocity law near the wall is well known from many years (von Kármán (1930)) for what concerns the flow in smooth pipes (Nikuradse (1932), Reichardt (1940)) and in the cases of turbulent boundary layers with moderate pressure gradient (Klebanoff (1952), Klebanoff (1955), Smith & Walker (1958)). This law, usually expressed as

$$U^+ = y^+, \quad y^+ \leq 5, \quad (1)$$

$$U^+ \simeq A \ln y^+ + B, \quad 5 < y^+ < 30, \quad (2)$$

$$U^+ = \frac{1}{k} \ln y^+ + C, \quad y^+ > 30, \quad (3)$$

gives the average velocity in different regions near the wall, where $y^+ = yU_T/\nu$ and $U^+ = U/U_T$ are the dimensionless normal coordinate and average velocity, and $U_T = \sqrt{\nu(\partial U/\partial y)_0}$ is the friction velocity.

These expressions and the values of A , B , C and k , seem to be universal properties of the flow, which do not depend on the Reynolds number. With reference to Fig. 1, these equations, obtained through considerations of dimensional analysis and of self-similarity, hold under the hypothesis of fully developed parallel flow along the streamwise direction x (von Kármán (1930)). Specifically, Eq. (1), being the direct consequence of the wall boundary condition and of the definition of U_T , expresses the velocity distribution in the laminar sub-layer LL, a domain adjacent to the wall where the effects of the viscosity are dominant. Equation (3) holds in the turbulent region TR, a zone of non-homogeneous turbulence, where the Taylor scale Reynolds number is high and variable with y . Into Eq. (3), k is the von Kármán constant which according to von Kármán (1930) and in line with Eq. (3), can also be expressed as

$$k = - \lim_{y^+ \rightarrow y_e^+} \frac{\frac{d^2 U^+}{dy^{+2}}}{\left(\frac{dU^+}{dy^+}\right)^2} \quad (4)$$

where y_e^+ defines the lower limit of the turbulent region. As far as Eq. (2) is concerned, this describes the velocity variations in the buffer layer BL, an intermediate zone between TR and LL in which viscous and inertia forces are comparable as order of magnitude.

k and A are directly related to the velocity variations along y , whereas C and B express the order of magnitude of U with respect to the friction velocity in case of smooth wall. These constants are dimensionless free parameters which can not be theoretically calculated (Landau & Lifshitz (1959)), therefore several experiments dealing with their determination were carried out (Fernholz & Finleyt (1996), Zagarola & Smits (1998)), and scaling law similarities were proposed to justify Eq. (3) (Barenblatt (1993)). These

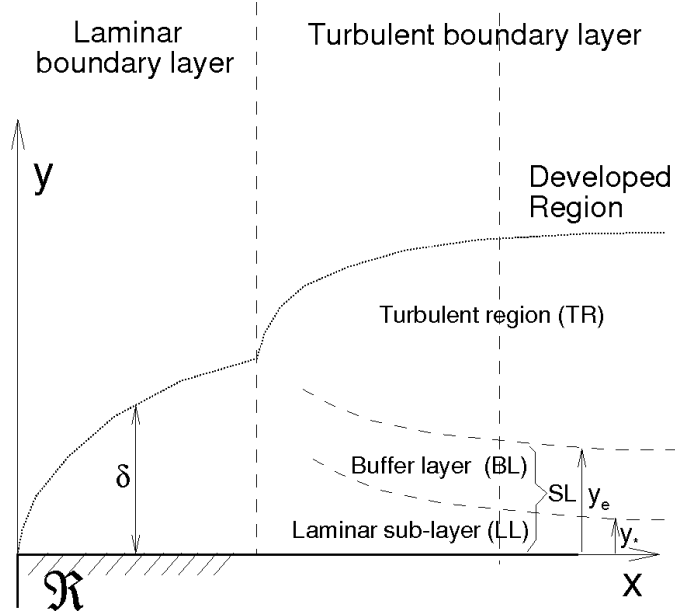


Figure 1: Schematic of the boundary layer.

constants can be also identified through the elaboration of the results of direct numerical simulations of the Navier-Stokes equations (Spalart (1988), Fernholz & Finley (1996) and references therein). From the various sources of the literature, $k = 0.36 \div 0.44$, $C = 4.5 \div 7.5$, $A \simeq 5 \div 6$ and $B \simeq -5 \div -3$.

Another law of the wall, also valid under the conditions of parallel flow and self-similarity, and that seems to show properties of universality, pertains the Reynolds stress $\langle u_x u_y \rangle$

$$-\langle uv \rangle^+ = g(y^+), \quad \langle uv \rangle^+ = \langle u_x u_y \rangle / U_T^2 \quad (5)$$

This behaves like $g \approx y^{+3}$ very near the wall, and at about $y^+ = 5$, exhibits an inflection point whose coordinate provides the order of magnitude of the laminar sub-layer thickness. The value $-\langle uv \rangle^+ = 0.5$, achieved in the buffer layer at about $y^+ = 10 \div 13$, corresponds to the maximum turbulent energy production due to the mean flow, whereas far from the wall $g \approx 1$ (Hinze (1975), Tennekes & Lumley (1972)).

Although all these distributions seem to be universal laws, according to Landau & Lifshitz (1959) and Fernholz & Finley (1996), there is not an adequate theory based on physical conjectures, other than the direct numerical simulations of the Navier Stokes equations, which leads to the calculation

of the free parameters of these laws. This represents the main motivation of the present work, whose purpose is to apply the finite scale Lyapunov theory, described in de Divitiis (IJES 2011), to determine the wall velocity distribution and its parameters.

The first part of the work summarizes the main results of de Divitiis (IJES 2011), which deals with the steady homogeneous turbulence with uniform velocity gradient, and then shows that such results can be extended also to the non-homogeneous turbulence with shear rate. We give a reasonable demonstration that the equation of the correlation function obtained by de Divitiis (IJES 2011) can hold also in the case of non-homogeneous turbulence, in particular in the turbulent zone of the boundary layer. This result allows to express, in the turbulent region, the several quantities such as dimensionless velocity gradient and turbulent kinetic energy in function of the local Taylor-scale Reynolds number, whereas in the laminar sub-layer and in the buffer region, adequate variation laws of the variables are assumed, which satisfy the wall boundary conditions and match the values of the variables in the adjacent domains.

The analysis gives the von Kármán constant in terms of the variables at the edge of the turbulent region, and provides a logarithmic velocity profile in TR, different from Eq. (3), which exhibits free parameters (k included). The knowledge of the statistical properties of the spanwise correlation function leads to the estimation of the variables at y_e^+ and thus to identify k and the other parameters. Therefore, the von Kármán is here theoretically obtained, resulting to be about 0.4, in line with experiments and numerical simulations, and the other results, in agreement with the several data from the literature, show that the finite scale Lyapunov theory can be an adequate tool for studying the wall turbulence.

2. Resume

This section summarizes the main results of de Divitiis (IJES 2011) which regards the steady homogeneous turbulence in the presence of an average velocity gradient $\nabla_{\mathbf{x}}\mathbf{U}$. There, the author, applying the Finite-scale Lyapunov theory (de Divitiis (2010)) and the Liouville theorem, proposes the following evolution equation of the pair distribution function $F^{(2)}$ of fluid velocities in

case of arbitrary flow

$$\frac{\partial F^{(2)}}{\partial t} + \nabla_{\mathbf{x}} F^{(2)} \cdot \mathbf{v} + \nabla_{\mathbf{x}'} F^{(2)} \cdot \mathbf{v}' = \lambda(r) \left(F_0^{(2)} - F^{(2)} \right) - J_D \quad (6)$$

where $F_0^{(2)}$ is the pair distribution function of the isotropic turbulence which exhibits the same momentum and kinetic energy of $F^{(2)}$, whereas $-J_D$ represents the rate of $F^{(2)}$ caused by the rate of the turbulent kinetic energy. $\lambda(r)$ is the finite-scale Lyapunov exponent associated to the finite-scale $r = |\mathbf{x}' - \mathbf{x}|$, and \mathbf{v} and \mathbf{v}' are the fluid velocities calculated at \mathbf{x} and \mathbf{x}' respectively. In case of homogeneous turbulence, de Divitiis (IJES 2011) shows that the steady distribution function reasonably tends to a quantity which depends upon $F_0^{(2)}$, $\nabla_{\mathbf{x}} \mathbf{U}$ and λ

$$F^{(2)} = F_0^{(2)} + \frac{1}{\lambda(r)} \left(\frac{\partial F_0^{(2)}}{\partial v_j} \frac{\partial U_j}{\partial x_p} v_p + \frac{\partial F_0^{(2)}}{\partial v'_j} \frac{\partial U_j}{\partial x_p} v'_p \right) \quad (7)$$

The velocity correlation tensor $R_{ki} = \langle u_i u'_j \rangle$ is then calculated, by definition

$$R_{ki} = \int_v \int_{v'} F^{(2)} u_k u'_i d^3 u d^3 u' = R_{ki0} - \frac{1}{\lambda} \left(\frac{\partial U_k}{\partial x_p} R_{pi0} + \frac{\partial U_i}{\partial x_q} R_{kq0} \right) \quad (8)$$

where R_{ki0} is the second order velocity correlation tensor associated to the isotropic turbulence (von Kármán & Howarth (1938), Batchelor (1953))

$$R_{ki0}(\mathbf{r}) = u^2 \left((f - g) \frac{r_k r_i}{r^2} + g \delta_{ki} \right) \quad (9)$$

being f and $g = f + 1/2 r \partial f / \partial r$ longitudinal and lateral velocity correlation functions, respectively, and $\mathbf{u} = \mathbf{v} - \mathbf{U}$ is the fluctuating velocity. For $r = 0$, Eq. (8) provides the expression of the Reynolds stresses in function of $\nabla_{\mathbf{x}} \mathbf{U}$

$$\langle u_k u_i \rangle = u^2 \left(\delta_{ki} - \frac{1}{\Lambda} \left(\frac{\partial U_k}{\partial x_i} + \frac{\partial U_i}{\partial x_k} \right) \right) \quad (10)$$

Equation (10) is a Boussinesq closure of the Reynolds stress, being $u = \langle u_i u_i \rangle / 3$, $\langle . \rangle$ indicates the average calculated on the ensemble of the fluid velocity, and $\Lambda = \lambda(0)$ is the maximal Lyapunov exponent. The Lyapunov theory presented in de Divitiis (2010) shows that $\Lambda = u / \lambda_T$, being λ_T the Taylor scale.

Next, the condition of steady flow leads to the following ordinary differential equation for f de Divitiis (IJES 2011)

$$\sqrt{\frac{1-f}{2}} \frac{df}{d\hat{r}} + \frac{2}{R_T} \left(\frac{d^2 f}{d\hat{r}^2} + \frac{4}{\hat{r}} \frac{df}{d\hat{r}} \right) + \frac{10}{R_T} \frac{f \hat{r}}{\sqrt{2(1-f)}} = 0 \quad (11)$$

whose boundary conditions can be reduced to the following conditions of f in the origin $r = 0$

$$f(0) = 1, \quad \frac{df(0)}{d\hat{r}} = 0 \quad (12)$$

where $\hat{r} = r/\lambda_T$, $d^2 f/d\hat{r}^2(0) \equiv -1$, and $R_T = u\lambda_T/\nu$ is the Taylor scale Reynolds number. Thus, Eqs. (11) and (12) express an initial condition problem whose initial condition is given by Eqs. (12). The same steady condition gives the relationship between $\nabla_{\mathbf{x}}\mathbf{U}$, Λ and R_T , which represents the equation of the kinetic energy in the case of steady homogeneous turbulence

$$\frac{S}{\Lambda^2} = \frac{15}{R_T} \quad (13)$$

where S is related to $\nabla_{\mathbf{x}}\mathbf{U}$

$$S = \frac{\partial U_i}{\partial x_k} \left(\frac{\partial U_k}{\partial x_i} + \frac{\partial U_i}{\partial x_k} \right) \quad (14)$$

The tensor R_{ik} is then calculated with Eq. (8) and (9), by means of f .

de Divitiis (IJES 2011) studies some of the properties of Eqs. (11)-(12), and shows that their solutions behave like $f - 1 \approx r$ in a given interval of r . Accordingly, the statistical moments of velocity difference are $\langle \Delta u^n \rangle \approx r^{n/2}$ for $n < 5$, and the energy spectrum $E(\kappa) \approx k^{-2}$ in the inertial subrange.

Equation (6) was obtained for an arbitrary flow, whereas the other equations were derived under the assumption of steady homogeneous turbulence with a given average velocity gradient. The successive sections show that these equations can be applied also in the case of non-homogeneous turbulence where λ_T and u vary with the space coordinates. Thus, the method is here applied to the turbulent region of the fully developed boundary layer, where the several quantities vary with the wall normal coordinate.

3. Analysis

An evolution equation for the velocity correlation is determined, in the case of non-homogeneous turbulence with a nonzero average velocity gradient.

The fluid velocity, measured in the reference frame \mathfrak{R} , is $\mathbf{v} = \mathbf{U} + \mathbf{u}$, where $\mathbf{U} \equiv (U_x, U_y, U_z)$ and $\mathbf{u} \equiv (u_x, u_y, u_z)$ are, average and fluctuating velocity, respectively. The velocity correlation tensor is $R_{ij} = \langle u_i u'_j \rangle$, where u_i and u'_j are the velocity components of \mathbf{u} calculated at \mathbf{x} and $\mathbf{x}' = \mathbf{x} + \mathbf{r}$, and \mathbf{r} is the separation distance. As the analysis is finalized to the description of the turbulent region of the developed boundary layer, and since there the effects of the spatial variations of $\nabla_{\mathbf{x}} \mathbf{U}$ are orders of magnitude much smaller than those caused by $\nabla_{\mathbf{x}} \mathbf{U}$ (von Kármán (1930), Nikuradse (1932)), the velocity gradient is assumed to be a function of \mathbf{x} alone, being $\nabla_{\mathbf{x}} \mathbf{U} = \nabla_{\mathbf{x}} \mathbf{U}'$.

In order to determine the evolution equation of R_{ij} , the Navier-Stokes equations are written for the fluctuating velocity in the points \mathbf{x} and \mathbf{x}' . The evolution equation of R_{ij} is determined by multiplying first and second equation by u'_j and u_i , respectively, summing the so obtained equations, and calculating the average on the statistical ensemble (von Kármán & Howarth (1938), Batchelor (1953))

$$\frac{\partial R_{ij}}{\partial t} = T_{ij} + P_{ij} + 2\nu \nabla^2 R_{ij} - \frac{\partial U_i}{\partial x_k} R_{kj} - \frac{\partial U_j}{\partial x_k} R_{ik} + \frac{\partial R_{ij}}{\partial r_k} (U_k - U'_k) + \Gamma_{ij} \quad (15)$$

being

$$T_{ij}(\mathbf{x}, \mathbf{r}) = \frac{\partial}{\partial r_k} \langle u_i u'_j (u_k - u'_k) \rangle, \quad P_{ij}(\mathbf{x}, \mathbf{r}) = \frac{1}{\rho} \frac{\partial \langle p u'_j \rangle}{\partial r_i} \quad (16)$$

and p is the fluctuating pressure. The quantities of Eq. (15), which in turn depend on \mathbf{r} , due to non-homogeneity depend also on \mathbf{x} , and Eq. (15) includes an additional term with respect to the homogeneous turbulence, represented by $\Gamma_{ij}(\mathbf{x}, \mathbf{r})$ (Oberlack (1997))

$$\begin{aligned} \Gamma_{ij}(\mathbf{x}, \mathbf{r}) = & -\frac{1}{\rho} \frac{\partial \langle p u'_j \rangle}{\partial x_i} - \frac{\partial}{\partial x_k} \langle u_i u_k u'_j \rangle - \frac{1}{\rho} \frac{\partial \langle p' u_i \rangle}{\partial r_j} + \nu \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} \\ & - 2\nu \frac{\partial^2 R_{ij}}{\partial r_k \partial x_k} \end{aligned} \quad (17)$$

which provides the non-homogeneity of the different terms of correlation.

Making the trace of Eq. (15), we obtain the following scalar equation

$$\frac{\partial R}{\partial t} = \frac{1}{2}H + 2\nu\nabla^2 R - \frac{\partial U_i}{\partial x_k} R_{ik}^S + \frac{\partial R}{\partial r_k} (U_k - U'_k) + \Gamma \quad (18)$$

being R_{ik}^S is the symmetric part of R_{ik} and

$$R = \frac{R_{ii}}{2}, \quad \Gamma = \frac{\Gamma_{ii}}{2} \quad (19)$$

$R(\mathbf{x}, 0)$ gives the turbulent kinetic energy, $H \equiv T_{ii}$ provides the mechanism of energy cascade (von Kármán & Howarth (1938), Batchelor (1953)), $P_{ii} \equiv 0$ expresses the fluid incompressibility whereas Γ arises from the non-homogeneity of the flow.

Now, a scalar equation for describing the main properties of the velocity correlation is determined. To this end, R_{ij} , H and \mathbf{U} are decomposed into an even function of $r \equiv |\mathbf{r}|$ (here called spherical part), plus the remaining term:

$$\begin{aligned} R_{ij}(\mathbf{x}, \mathbf{r}) &= \hat{R}_{ij}(\mathbf{x}, r) + \Delta R_{ij}(\mathbf{x}, \mathbf{r}) \\ H(\mathbf{x}, \mathbf{r}) &= \hat{H}(\mathbf{x}, r) + \Delta H(\mathbf{x}, \mathbf{r}) \\ \mathbf{U}' - \mathbf{U} &= \hat{\mathbf{U}}(\mathbf{x}, r) + \Delta \mathbf{U}(\mathbf{x}, \mathbf{r}) \end{aligned} \quad (20)$$

where \hat{F} is the spherical part of the generic quantity F , defined as

$$\begin{aligned} \hat{F}(\mathbf{x}, r) &= \frac{1}{6} (F(\mathbf{x}, r, 0, 0) + F(\mathbf{x}, 0, r, 0) + F(\mathbf{x}, 0, 0, r)) \\ &+ \frac{1}{6} (F(\mathbf{x}, -r, 0, 0) + F(\mathbf{x}, 0, -r, 0) + F(\mathbf{x}, 0, 0, -r)) \end{aligned} \quad (21)$$

and $\Delta R_{ij}(\mathbf{x}, \mathbf{0}) = \Delta H(\mathbf{x}, \mathbf{0}) = 0$. Therefore, the Fourier transform of \hat{R} identifies the part of the energy spectrum depending upon $\boldsymbol{\kappa}^2$. As the consequence of this decomposition, R and ΔR satisfy the equations

$$\frac{\partial \hat{R}}{\partial t} = \frac{\hat{H}}{2} + 2\nu \left(\frac{\partial^2 \hat{R}}{\partial r^2} + \frac{2}{r} \frac{\partial \hat{R}}{\partial r} \right) - \hat{G} \quad (22)$$

$$\frac{\partial \Delta R}{\partial t} = \frac{\Delta H}{2} + 2\nu \nabla^2 \Delta R - \Delta G \quad (23)$$

in which Eq. (23) is obtained as the difference between Eqs. (18) and (22), and

$$\hat{G} = \frac{\partial U_i}{\partial x_k} \hat{R}_{ki}^S + \hat{G}_0, \quad \Delta G = \frac{\partial U_i}{\partial x_k} \Delta R_{ki}^S - \Delta \left(\frac{\partial R}{\partial r_k} (U_k - U'_k) \right) - \Delta \Gamma \quad (24)$$

and \hat{G}_0 represents the spherical part of $-\partial R / \partial r_k (U_k - U'_k) - \Gamma$. It is worth to remark that Eq. (22) formally coincides with the equation obtained in de Divitiis (IJES 2011), with the difference that here, because of non-homogeneity, the quantities appearing into Eq. (22) depend also on \mathbf{x} .

The turbulence is here studied using Eq. (22) alone, whereas $R_{ij}(\mathbf{x}, \mathbf{r})$ will be determined in function of $\nabla_{\mathbf{x}} \mathbf{U}$, by means of a proper statistical analysis of the two-points velocity correlation.

4. Pair distribution function

This section analyses the non-homogeneous turbulence through the pair distribution function, taking into account that λ_T and $\langle u_i u_j \rangle$ vary with the spatial coordinates, whereas $\nabla_{\mathbf{x}} \mathbf{U}$ is considered to be an assigned quantity. To study this, consider now the pair distribution function of the fluid velocity

$$F^{(2)}(\mathbf{v}, \mathbf{v}'; \mathbf{x}, \mathbf{x}') = F_0^{(2)}(\mathbf{v}, \mathbf{v}'; \mathbf{x}, \mathbf{x}') + \phi^{(2)}(\mathbf{v}, \mathbf{v}'; \mathbf{x}, \mathbf{x}') \quad (25)$$

where $F^{(2)}$ obeys to Eq. (6), and $\phi^{(2)}$, representing the deviation from the isotropic turbulence, satisfies, at each instant, the following equations (de Divitiis (IJES 2011))

$$\begin{aligned} \int_v \int_{v'} \phi^{(2)} du^3 du'^3 &= 0, \\ \int_v \int_{v'} \phi^{(2)} \mathbf{u} du^3 du'^3 &= 0, \\ \int_v \int_{v'} \phi^{(2)} \mathbf{u} \cdot \mathbf{u} du^3 du'^3 &= 0. \end{aligned} \quad (26)$$

Equations (26) state that momentum and kinetic energy associated to $F_0^{(2)}$ and $F^{(2)}$ are equal each other, respectively. The analysis supposes also that

all the dimensionless statistical moments of $F_0^{(2)}$ are constant in all the points of the fluid domain, therefore the functional form of $F_0^{(2)}$ is assumed to be

$$F_0^{(2)}(\mathbf{v}, \mathbf{v}'; \mathbf{x}, \mathbf{x}') = F_0^{(2)} \left(\frac{\mathbf{v} - \mathbf{U}(\mathbf{x})}{u(\mathbf{x})}, \frac{\mathbf{v}' - \mathbf{U}(\mathbf{x}')}{u(\mathbf{x}')} \right) \quad (27)$$

and its gradients are calculated in functions of the spatial derivatives of \mathbf{U} and u

$$\frac{\partial F_0^{(2)}}{\partial x_k} = -\frac{\partial F_0^{(2)}}{\partial v_j} \left(\frac{\partial U_j}{\partial x_k} + \frac{u_j}{u} \frac{\partial u}{\partial x_k} \right), \quad \frac{\partial F_0^{(2)}}{\partial x'_k} = -\frac{\partial F_0^{(2)}}{\partial v'_j} \left(\frac{\partial U_j}{\partial x_k} + \frac{u'_j}{u'} \frac{\partial u'}{\partial x'_k} \right) \quad (28)$$

where, as before, $\nabla_{\mathbf{x}} \mathbf{U} = \nabla_{\mathbf{x}'} \mathbf{U}'$. Substituting Eqs.(25) and (28) into Eq. (6), we obtain the following relationship between $F_0^{(2)}$ and $\phi^{(2)}$

$$\begin{aligned} \lambda \phi^{(2)} = & -J_D - \left(\frac{\partial F_0^{(2)}}{\partial t} + \frac{\partial \phi^{(2)}}{\partial t} + \frac{\partial \phi^{(2)}}{\partial x_p} v_p + \frac{\partial \phi^{(2)}}{\partial x'_p} v'_p \right) \\ & + \left(\frac{\partial F_0^{(2)}}{\partial v_j} \frac{v_p u_j}{u} \frac{\partial u}{\partial x_k} \right) + \left(\frac{\partial F_0^{(2)}}{\partial v'_j} \frac{v'_p u'_j}{u'} \frac{\partial u'}{\partial x'_k} \right) \\ & + \left(\frac{\partial F_0^{(2)}}{\partial v_j} v_p + \frac{\partial F_0^{(2)}}{\partial v'_j} v'_p \right) \frac{\partial U_j}{\partial x_p} \end{aligned} \quad (29)$$

As the last term of Eq. (29) identically satisfies Eqs. (26), these latter are written in the form

$$\begin{aligned} \int_v \int_{v'} \left[\begin{array}{c} 1 \\ \mathbf{u} \\ \mathbf{u} \cdot \mathbf{u} \end{array} \right] & \left(\frac{\partial F_0^{(2)}}{\partial t} + \frac{\partial \phi^{(2)}}{\partial x_p} v_p + \frac{\partial \phi^{(2)}}{\partial x'_p} v'_p - \frac{\partial F_0^{(2)}}{\partial v_j} \frac{v_p u_j}{u} \frac{\partial u}{\partial x_k} \right. \\ & \left. - \frac{\partial F_0^{(2)}}{\partial v'_j} \frac{v'_p u'_j}{u'} \frac{\partial u'}{\partial x'_k} + J_D \right) du^3 du'^3 \equiv 0 \end{aligned} \quad (30)$$

A sufficient condition for satisfying these five equations is that the integrand of Eq. (30) vanishes

$$\frac{\partial F_0^{(2)}}{\partial t} + \frac{\partial \phi^{(2)}}{\partial x_p} v_p + \frac{\partial \phi^{(2)}}{\partial x'_p} v'_p - \frac{\partial F_0^{(2)}}{\partial v_j} \frac{v_p u_j}{u} \frac{\partial u}{\partial x_k} - \frac{\partial F_0^{(2)}}{\partial v'_j} \frac{v'_p u'_j}{u'} \frac{\partial u'}{\partial x'_k} + J_D \equiv 0 \quad (31)$$

Assuming that Eq. (31) is true, and taking into account Eq. (29), $F^{(2)}$ is given by Eq. (7) which does not depend on the gradient of u and u' , as in the case of the homogeneous turbulence. Really, $F^{(2)}$ changes starting from an arbitrary initial condition, therefore Eq. (7) represents an approximation which can be considered to be valid far from the initial condition. As the consequence, also the spherical part of the correlation tensor, \hat{R}_{ij} , here obtained

$$\hat{R}_{ki} = \hat{R}_{ik}^S = \frac{u^2}{3} \left(3f + \frac{\partial f}{\partial r} r \right) \left(\delta_{ki} - \frac{1}{\lambda} \left(\frac{\partial U_k}{\partial x_i} + \frac{\partial U_i}{\partial x_k} \right) \right) \quad (32)$$

coincides with the expression given in de Divitiis (IJES 2011). Thus, substituting Eq. (32) and into Eq. (22) and following the analytical procedure of de Divitiis (IJES 2011), we found that f obeys to Eq. (11) also in the present case, and Λ is related to $\nabla_{\mathbf{x}} \mathbf{U}$ and R_T through Eq. (13). The difference with respect to de Divitiis (IJES 2011) is that, here the turbulence is non-homogeneous, thus $R_T = R_T(\mathbf{x})$, and f also depends on \mathbf{x} , being $f = f(\mathbf{x}, r)$.

5. Analysis of the boundary layer

Here, the steady turbulent boundary layer with a moderate pressure gradient is analyzed, assuming that the flow is fully developed along the streamwise direction. To this purpose, return to Fig. 1, and consider only the developed region of the flow. In the figure, \Re is the wall frame of reference, x and y are, respectively, the streamwise direction and the coordinate normal to the wall, whereas z is the spanwise coordinate.

In the developed region, u , $\langle u_x u_y \rangle$ and λ_T change with y , and in the laminar sublayer U_x and u are both about proportional to y . In LL, BL and at the beginning of TR, the correlation scale of velocity is proportional to the distance from the wall, and vanishes for $y = 0$, being

$$\lambda_T = \left(\frac{\partial \lambda_T}{\partial y} \right)_0 y + \dots \quad (33)$$

where $(\partial \lambda_T / \partial y)_0 = O(1)$, whereas the velocity fluctuations follow the Navier-Stokes equations and satisfy the wall boundary conditions

$$u_x = \frac{\partial u_x}{\partial y}(0)y + \dots, \quad u_y = \frac{1}{2} \frac{\partial^2 u_y}{\partial y^2}(0)y^2 + \dots, \quad u_z = \frac{\partial u_z}{\partial y}(0)y + \dots \quad (34)$$

In case of fully developed flow along x , that is, parallel flow assumption ($\partial/\partial y \gg \partial/\partial x$), the continuity and momentum equations of the mean flow are (Schlichting (2004))

$$\frac{\partial U_y}{\partial y} = 0, \quad (35)$$

$$\begin{aligned} \frac{\partial}{\partial y} \langle u_x u_y \rangle + \frac{1}{\rho} \frac{\partial P}{\partial x} - \nu \frac{\partial^2 U_x}{\partial y^2} &= 0, \\ \frac{\partial}{\partial y} \langle u_y^2 \rangle + \frac{1}{\rho} \frac{\partial P}{\partial y} &= 0 \end{aligned} \quad (36)$$

where P is the average pressure, whereas the equation of the turbulent kinetic energy reads as

$$\frac{\partial}{\partial y} \left\langle u_y \left(\frac{p}{\rho} + \frac{u_j u_j}{2} \right) \right\rangle + \langle u_x u_y \rangle \frac{\partial U_x}{\partial y} + \nu \left\langle \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i} \right\rangle = 0 \quad (37)$$

The non-homogeneity is responsible for the first term of Eq. (37), whereas second and third terms represent, respectively, the energy production due to the average motion, and the dissipation.

Because of the parallel flow assumption, in all these equations U_x , $\langle u_i u_j \rangle$ are functions of y alone. From Eq. (35) and taking into account the boundary conditions, $U_y \equiv 0$, whereas Eqs. (36) give the following first integrals

$$\frac{P(x, y)}{\rho} = Fx + H - \langle u_y^2 \rangle \quad (38)$$

$$\langle u_x u_y \rangle - \nu \frac{dU_x}{dy} = -Fy - U_T^2 \quad (39)$$

being $F = 1/\rho \partial P/\partial x$ and H is a proper constant proportional to the average pressure at $x = 0$. Into Eq. (39), the boundary layer approximation and the hypothesis of moderate average pressure gradient along x provide that $U_T^2 \gg |Fy|$ in all the regions, therefore introducing the dimensionless variables $U^+ = U_x/U_T$ and $y^+ = yU_T/\nu$, Eq. (39) reads as

$$\frac{dU^+}{dy^+} = \frac{1}{1 + R_T} \quad (40)$$

Equation (40) is assumed to describe the average flow with moderate pressure gradient in the three regions of the boundary layers, when the parallel flow hypothesis is verified.

5.1. The Turbulent Region

This section studies the distribution of the different variables in the turbulent region TR, by means of the analysis seen in the sections 3 and 4.

First, observe that, in case of fully developed parallel flow, the effects of non-homogeneity in TR are much smaller than those related to energy production and dissipation (Tennekes & Lumley (1972)), thus the first term of Eq. (37) is here neglected with respect to the other ones, and Eq. (13) is recovered. This approximation, in agreement with the analysis seen in sect. 4, states that the kinetic energy production is balanced only by the dissipation in TR, and allows to express the several variables in terms of the local value of R_T (or dU_x/dy).

In order to calculate $u^+ = u/U_T$, dU_x/dy is eliminated between Eqs. (39) and (13), where $S = (dU_x/dy)^2$

$$u^+ = \frac{R_T^{3/4}}{15^{1/4}\sqrt{1+R_T}} \quad (41)$$

Being $R_T = u^+ \lambda_T^+$, also λ_T^+ is in terms of R_T

$$\lambda_T^+ = (15R_T)^{1/4}\sqrt{1+R_T} \quad (42)$$

As these equations arise from Eq. (13) which holds in the turbulent region, Eqs.(41) and (42) describe the variations of u and λ_T only in TR.

From the comparison between Eqs. (40) and (4), $(dR_T/dy^+)_e$ identifies the von Kármán constant, here expressed taking into account that $R_T \equiv \lambda_T^+ u^+$

$$k \equiv \left(\frac{dR_T}{dy^+}\right)_e = \left(\frac{du^+}{dy^+}\right)_e \lambda_{Te}^+ + \left(\frac{d\lambda_T^+}{dy^+}\right)_e u_e^+ \quad (43)$$

where the subscript e denotes the values calculated at the edge of TR. Substituting Eqs. (41) and (42) into Eq. (43), we obtain the von Kármán constant in function of the variables at y_e^+

$$k = \frac{4}{15^{1/4}} \left(\frac{d\lambda_T^+}{dy^+}\right)_e \frac{\sqrt{1+R_{Te}}}{1+3R_{Te}} R_{Te}^{3/4} \quad (44)$$

It is worth to remark that this estimation of k requires the knowledge of R_{Te} and of $(d\lambda_T^+/dy^+)_e$, whereas does not need the assumption that U^+ is represented by a logarithmic profile.

Next, the dimensionless Prandtl's mixing scale l_p^+ is calculated from the definition of Prandtl's mixing length l_p

$$-\langle u_x u_y \rangle = \left| \frac{\partial U_x}{\partial y} \right| \frac{\partial U_x}{\partial y} l_p^2 \quad (45)$$

and Eqs. (10) and (13). This is

$$l_p^+ = \sqrt{R_T(1 + R_T)} \quad (46)$$

being $l_p^+ \simeq R_T$ for $R_T \gg 1$, and its derivative calculated at y_e^+ is also expressed in terms of R_{Te}

$$\left(\frac{dl_p^+}{dy^+} \right)_e = \frac{1 + 2R_{Te}}{2\sqrt{R_{Te}(1 + R_{Te})}} k \quad (47)$$

In view of Eq. (44), this derivative can be also expressed in function of $d\lambda_T^+/dy$

$$\left(\frac{dl_p^+}{dy^+} \right)_e = 2 \frac{1 + 2R_{Te}}{1 + 3R_{Te}} \left(\frac{R_{Te}}{15} \right)^{1/4} \left(\frac{d\lambda_T^+}{dy^+} \right)_e \quad (48)$$

This expression gives the link between the variations of the Taylor scale and of the Prandtl's length at the border of the turbulent region.

Following Eqs. (41) and (42), u^+ and λ_T^+ are functions of y^+ through the local value of R_T (or of dU^+/dy^+), therefore, the distribution of such quantities along y^+ require the knowledge of the function $R_T = R_T(y^+)$. This latter can be expressed as

$$R_T(y^+) = R_{Te} + \left(\frac{dR_T}{dy^+} \right)_e (y^+ - y_e^+) + O(y^+ - y_e^+)^2 \quad (49)$$

Now, in a range of y^+ where $O(y^+ - y_e^+)^2$ is negligible with respect to the other terms, dU^+/dy^+ is

$$\frac{dU^+}{dy^+} = \frac{1}{1 + R_{Te} + \left(\frac{dR_T}{dy^+} \right)_e (y^+ - y_e^+)} \quad (50)$$

U^+ exhibits there logarithmic law, obtained integrating Eq. (50) from y_e^+ to y^+

$$U^+ = \frac{1}{k} \ln \left(\frac{1 + R_{Te} + k(y^+ - y_e^+)}{1 + R_{Te}} \right) + U_e^+ \quad (51)$$

being $U_e^+ = U^+(y_e^+)$. This law is defined as soon as the parameters y_e^+ , k and U_e^+ are known. Equation (51) differs from the classical expression (3) and formally tends to Eq. (3) when $y^+ \rightarrow \infty$. Therefore, Eq. (51) can give values of U^+ sizably different from (3) for small y^+ . The comparison between these equations, for $y^+ \rightarrow \infty$ identifies C in terms of the variables at y_e

$$C = U_e^+ + \frac{1}{k} \ln \left(\frac{k}{1 + R_{Te}} \right) \quad (52)$$

As far as the Reynolds stress is concerned, it is expressed in function of R_T through Eqs. (39) and (40)

$$\langle uv \rangle^+ \equiv \frac{\langle u_x u_y \rangle}{U_T^2} = -\frac{R_T}{1 + R_T} \quad (53)$$

Observe that, the wall boundary conditions (34) state that $\langle uv \rangle^+ \approx y^{+3}$ near the wall, whereas Eq. (53) gives $\langle uv \rangle^+ \approx R_T \approx y^{+2}$. This disagreement is due to the fact that the expression of $\langle uv \rangle^+$ has not been derived from the correlation equation with $r = 0$, but arises from Eq. (39) eliminating dU_x/dy . This implies that Eq. (53) holds only in TR and BL, whereas in the laminar sub-layer a proper matching condition must be applied.

5.2. Matching Turbulent region - buffer layer

With reference to Fig. 1, the domains LL and BL constitute SL, a zone between wall and turbulent region. There, due to the presence of the wall, the analysis of sections 4 and 3 can not be applied. Therefore, the mean variables in SL are expressed in function of y^+ , taking into account the boundary conditions (34) and that λ_T follows Eq. (33), where it is assumed $(d\lambda_T^+/dy^+)_e = (d\lambda_T^+/dy^+)_0$. As the consequence, R_T , u^+ and λ_T^+ are supposed

to vary in SL according to

$$R_T = u^+ \lambda_T^+,$$

$$u^+ = u_e^+ \frac{y^+ + C_u y^{+2}}{y_e^+ + C_u y_e^{+2}}, \quad (0 < y^+ \leq y_e^+) \quad (54)$$

$$\lambda_T^+ = \lambda_{Te}^+ \frac{y^+}{y_e^+}, \quad (0 < y^+ \leq y_e^+),$$

where C_u is a constant given by

$$C_u = \frac{1}{y_e^+} \frac{2R_{Te} - k y_e^+}{k y_e^+ - 3R_{Te}} \quad (55)$$

Equations (54) and (55) provide the matching condition between SL and TR. Specifically, Eqs. (54) state that u^+ , R_T and λ_T^+ are continuous functions for $y^+ = y_e^+$, whereas Eq. (55) gives there the continuity of their derivatives. The average velocity is then calculated by quadrature, substituting the expression of R_T given by Eqs. (54), into Eq. (40), integrating this latter from 0 to y^+ , with $U^+(0) = 0$

$$U^+(y^+) = \int_0^{y^+} \frac{1}{1 + R_T(\eta^+)} d\eta^+ \quad (56)$$

where $U^+(y^+) \simeq y^+$ for small y^+ . The matching between SL and TR gives the value of U_e^+ .

$$U_e^+ = \int_0^{y_e^+} \frac{1}{1 + R_T(\eta^+)} d\eta^+ \quad (57)$$

Thus, Eq. (56) and (51) establish that both U^+ and dU^+/dy^+ are continuous for $y^+ = y_e^+$

5.3. Matching buffer layer - laminar sub-layer

As previously seen, the expression (53) of the Reynolds stress, valid in BL and TR, can not be applied in the laminar region. Since $\langle uv \rangle^+ \approx y^{+3}$ near the wall, the Reynolds stress in LL is approximated by

$$\langle uv \rangle^+ = \langle uv \rangle_*^+ \frac{y^{+3} + C_{uv} y^{+4}}{y_*^{+3} + C_{uv} y_*^{+4}}, \quad (0 < y^+ < y_*^+) \quad (58)$$

being C_{uv} a constant

$$C_{uv} = \frac{1}{y_*^+} \frac{3 - (d \ln \langle uv \rangle^+ / dy^+)_* y_*^+}{(d \ln \langle uv \rangle^+ / dy^+)_* y_*^+ - 4} \quad (59)$$

Equations (58) and (59) establish that the Reynolds stress and its derivative are continuous for $y^+ = y_*^+$, where the subscript $*$ indicates the value calculated at y_*^+ . This latter, obtained as the inflection point of $\langle uv \rangle^+$ (i.e. $d^2 \langle uv \rangle^+ / dy^{+2} = 0$) according to Eqs. (53) and (54), gives the dimensionless tickness of the laminar sub-layer.

6. Identification of the velocity law free parameters

The definition of the velocity law, requires the knowledge of the parameters which appear into Eqs. (44) and (51). In particular, the determination of k needs the values of R_{Te} and $(d\lambda_T^+ / dy^+)_e$. To identify these latter, we will proceed as follow.

First, observe that $\langle u_i u_j \rangle$ is a symmetric tensor which can be obtained from the diagonal tensor

$$\langle \mathbf{u}_0 \mathbf{u}_0 \rangle = \begin{bmatrix} \langle u_0^2 \rangle & 0 & 0 \\ 0 & \langle v_0^2 \rangle & 0 \\ 0 & 0 & \langle w_0^2 \rangle \end{bmatrix} \quad (60)$$

through an opportune rotation around $y \equiv y_0$, being $\langle u_0^2 \rangle$, $\langle v_0^2 \rangle$ and $\langle w_0^2 \rangle$ the velocity components standard deviations in the canonical frame, and ϑ is the angle of this rotation. Therefore, u and $\langle u_x u_y \rangle$ are in terms of the elements of $\langle \mathbf{u}_0 \mathbf{u}_0 \rangle$

$$u^2 = \frac{1}{3} (\langle u_0^2 \rangle + \langle v_0^2 \rangle + \langle w_0^2 \rangle) \quad (61)$$

$$\langle u_x u_y \rangle = (\langle u_0^2 \rangle - \langle v_0^2 \rangle) \frac{\sin 2\vartheta}{2}$$

Because of the parallel flow assumption and taking into account Eqs. (10) and (61), u and $\langle u_x u_y \rangle$ are related each other in such a way that

$$u^2 \geq |\langle u_x u_y \rangle| \quad (62)$$

Hence, Eq. (13) implies that

$$R_T \geq 15 \quad \text{or} \quad \frac{dU_x}{dy} \lambda_T \geq u \quad (63)$$

in the turbulent region. This limitation identifies the minimum value of R_T in TR, which is assumed to be

$$R_{Te} = 15. \quad (64)$$

To estimate $(d\lambda_T^+/dy^+)_e$, consider first the spanwise correlation function of the streamwise velocity components (that is $R_{11}(r_z) = \langle u_x(x, y, z)u_x(x, y, z + r_z) \rangle$). This can be calculated with Eq. (8), once f is known through Eqs. (11). Since $\partial U/\partial y$ leads to the development of coherent structures in the fluid, similar to streaks and caused by the stretching of the vortex lines along the streamwise direction (Kim et al. (1990)), we expect that $R_{11}(r_z)$ intersects the horizontal axis and remains negative for $r_z \rightarrow \infty$. This implies a wide distribution of spacings between the different streaky structures, whose mean value depends on R_T (Kim et al. (1990)). Accordingly, there exists a minimum spanwise distance r_z^* such that, for $r_z \in [0, r_z^*]$, $R_{11}(r_z)$ gives the necessary informations to describe the statistical properties of the correlation (Ventsel (1973)). According to the theory (Ventsel (1973)), if R_{11} monotonically tends to zero as $r_z \rightarrow \infty$ and $R_{11} < 0$, r_z^* can be estimated as the distance at which R_{11} is negative and exhibits an inflection point (second inflection point of the curve, see Fig. 2)

$$R_{11}(r_z^*) < 0, \quad \frac{\partial^2 R_{11}}{\partial r_z^2}(r_z^*) = 0, \quad \text{Condition 1} \quad (65)$$

Alternatively, r_z^* can be estimated as the intersection between the osculating parabola in the point \bar{r}_z where $R_{11} = \min$, and the horizontal axis, that is

$$R_{11}(\bar{r}_z) + \frac{1}{2} \frac{\partial^2 R_{11}}{\partial r_z^2}(\bar{r}_z)(r_z^* - \bar{r}_z)^2 = 0, \quad \text{Condition 2} \quad (66)$$

Since $R_{11}(r_z)$ is expressed in function of f through Eq. (8), this is related to the correlation functions associated to the other directions, thus r_z^* is representative also for the other coordinates x and y . As the result, the distance from the wall of a point of TR must be always greater than r_z^* (i.e.

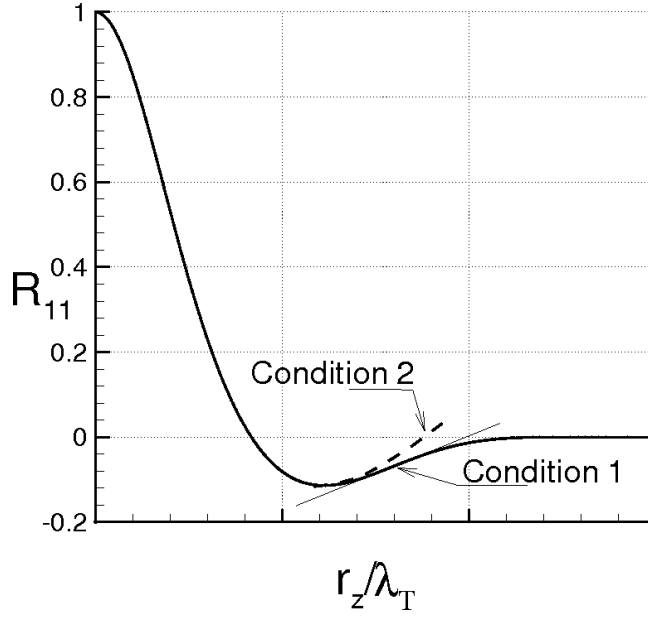


Figure 2: Schematic of the spanwise correlation function of u_x at the edge of the turbulent region and definition of Eqs. (65) and (66).

$r_z^* < y$). Hence, it is reasonable to assume that, at the edge of the turbulent domain

$$\begin{aligned} y_e &= r_z^* \\ R_{Te} &= 15 \end{aligned} \tag{67}$$

In order to calculate R_{11} , f is first obtained by solving Eqs. (11)-(12) which correspond to the following Cauchy's initial condition problem (de Divitiis (IJES 2011))

$$\begin{aligned} \frac{df}{d\hat{r}} &= F \\ \frac{dF}{d\hat{r}} &= -\frac{5f\hat{r}}{\sqrt{2(1-f)}} - \left(\frac{1}{2} \sqrt{\frac{1-f}{2}} R_T + \frac{4}{\hat{r}} \right) F \\ f(0) &= 1, \quad F(0) = 0 \end{aligned} \tag{68}$$

Hence, $R_{11}(r_z)$ is obtained with Eq. (8), and r_z^* is calculated for both the conditions (65) and (66). As the result, y_e^+ is given by

$$\frac{y_e}{\lambda_{Te}} = \frac{r_z^*}{\lambda_{Te}} \quad \text{where } R_T = R_{Te} \equiv 15 \quad (69)$$

and $(d\lambda_T^+/dy^+)_e$ is determined according to Eqs. (54)

$$\left(\frac{d\lambda_T^+}{dy^+} \right)_e = \frac{\lambda_{Te}^+}{y_e^+} \quad (70)$$

7. Results and Discussion

As the calculation of k and $d\lambda_T^+/dy^+$ needs the knowledge of the statistical properties of the velocity correlation, R_{11} is calculated for different values of R_T . To this purpose, f was first determined by solving numerically Eqs. (68), by means of the fourth-order Runge-Kutta method. The calculation was carried out for $R_T = 15, 20, 40, 60$ and 80 , where these Reynolds numbers correspond to several distances from the wall in the turbulent region. To obtain a good accuracy of the solutions, the step of integration is chosen to be equal to $1/40$ of the estimated Kolmogorov scale l_K , where $l_K/\lambda_T = 1/15^{1/4}/\sqrt{R_T}$ (Batchelor (1953)). The results are given in Fig. 3 (a) which shows f in terms of r , where the bold line represents the correlation function at the edge of TR ($R_T = 15$). This latter exhibits the ratio (integral scale)/(Taylor scale) quite similar to that of a gaussian centered in the origin, whereas in the other cases, this ratio increases with R_T , in agreement with the analysis of de Divitiis (IJES 2011). A more detailed analysis about the changing of f and of the corresponding energy spectrum $E(\kappa)$ with R_T is reported in de Divitiis (IJES 2011).

The spanwise correlation function $R_{11}(r_z)$ is then calculated with Eq. (8), and is represented in Fig. 3 (b) for the same values of R_T . From these data, the edge of TR, $y_e^+ \equiv r_z^*$, is calculated for both the conditions 1 and 2.

The von Kármán constant and all the others parameters are shown in the table 1. The table reports also the values of the free parameters A , B and C associated to the classical wall laws (1)-(3), which are here calculated through the identification with Eqs. (51) and (56). In particular, C is calculated with Eq. (52), whereas A , which pertains the velocity law in the

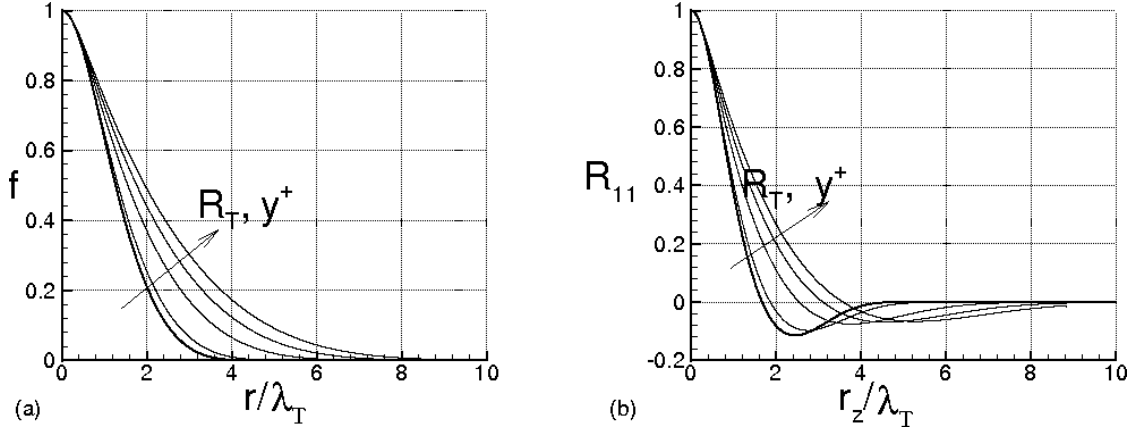


Figure 3: (a) Longitudinal correlation function associated to \hat{R}_{ij} . (b) spanwise correlation function of u_x , for different Taylor scale Reynolds number. The bold lines are calculated for $R_T = 15$.

buffer layer, is identified as the slope $dU^+(y^+)/d \ln y^+$ of Eq. (56) where $d^2U^+(y^+)/d \ln y^{+2} = 0$, and B is consequently determined. The values of k and C are compared with the results given by different authors.

For both the conditions, $y_e^+ \approx 50$ and $U_e^+ \approx 15$ and this corresponds to a difference with respect to the classical data of Nikuradse (1932) and Reichardt (1940), which is less than 2 %. As far as the Prandtl's length is concerned, according to Eq. (46), it varies quite similarly to R_T , and its derivative, calculated for $y^+ = y_e^+$, is slightly greater than k (see Eq. (47))

$$\left(\frac{dl_p^+}{dy^+} \right)_e = 1.34782... \left(\frac{d\lambda_T^+}{dy^+} \right)_e \simeq 1.00052 k \quad (71)$$

Its specific values, shown in the table, are in excellent agreement with the classical results. For what concerns A , B and C , their values agree quite well the experiments, expecially for what concerns C and A .

Once the free parameters are identified, R_T is calculated in function of y^+ through Eq. (49) with $O(y^+ - y_e^+)^2 \rightarrow 0$ and (54), for both the conditions (see Fig. 4), and this corresponds to consider only the logarithmic profile of U^+ . Figure 5 shows $U^+(y^+)$ calculated with Eq. (51) and (56). It is apparent that the results agree very well with the experimental data of different authors (Nikuradse (1932), Klebanoff (1952), Reichardt (1940)) and with the formulas proposed by Spalding (1961) and Musker (1979), with an error which does not never exceed 5 % in the interval of y^+ of the figure.

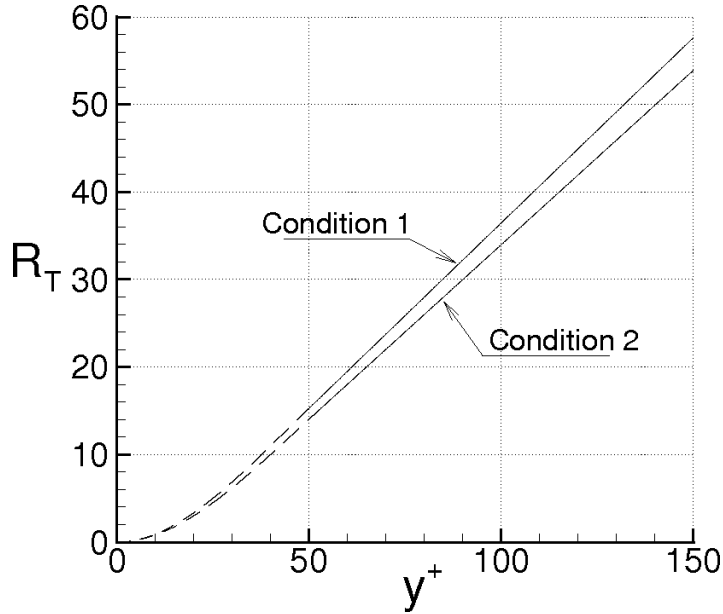


Figure 4: Taylor scale Reynolds number in terms of y^+ : SL dashed lines, TR continuous lines

In particular, the condition 1 (Eq. (65)) provides a maximum difference between present results and the data of Nikuradse and Reichardt, less than 3 % in this interval.

The other variables are represented in Fig. 6. The dimensionless Taylor scale (Fig. 6 a), linear in the laminar sub-layer and in the buffer region, remains a rising function of y^+ in the turbulent zone and exhibits, there, a slope which decreases slightly with y^+ , whereas the Lyapunov exponent (Fig. 6 b), defined only in TR, is represented by a monotonically decreasing function of y^+ . This latter, being proportional to the square root of the turbulent dissipation rate ($\Lambda = u/\lambda_T$), agrees with the data of the different experiments (Fernholz & Finleyt (1996)), at least where U^+ exhibits logarithmic profile.

The square root of the kinetic energy and the Reynolds stress are shown in Fig. 6 (c) and (d). These vary in TR following Eqs. (41)-(53) and are monotonically rising functions of y^+ . As seen, $u \approx y^+$ and $\langle uv \rangle^+ \approx y^{+3}$ in the laminar sub-layer, whereas the Reynolds stress calculated with Eq. (53) in the buffer region, shows an inflection point y_*^+ which represents the separation element between LL and BL. For $y^+ > y_*^+$, $-\langle uv \rangle^+$ rises with y^+ until to reach the turbulent region where is about constant and equal to the unity. More

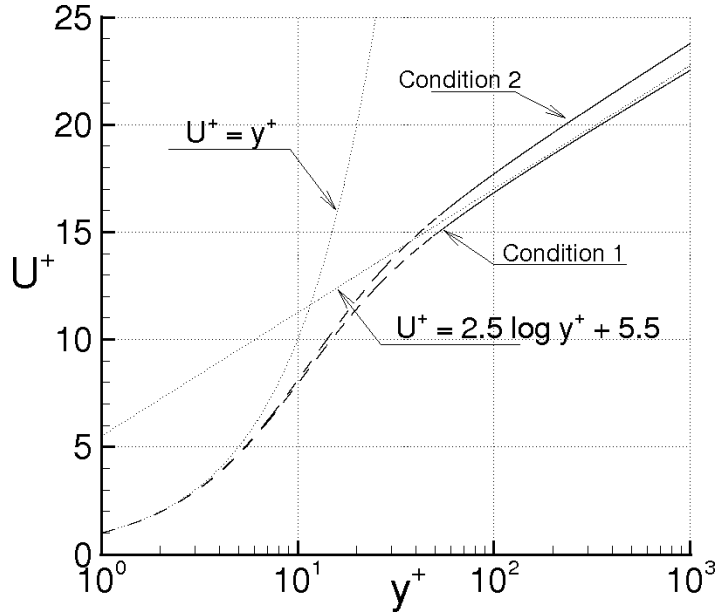


Figure 5: Dimensionless average velocity profile: SL dashed lines, TR continuous lines

in detail, Fig. 7 shows $-\langle uv \rangle^+$ in an enlarged region which includes LL and part of BL. The distance y_*^+ , represented by F_1 and F_2 is $y_*^+ = 5.734$ and 6.097 , in line with the order of magnitude of the laminar sub-layer thickness, and this is achieved at about $-\langle uv \rangle^+ = 0.24$ in the two cases. Next, the value $-\langle uv \rangle^+ = 0.5$ is obtained in the buffer layer for $y^+ \simeq 10.5$ and 11.2 , values in very good agreement with the classical results (Tennekes & Lumley (1972), Hinze (1975)). This last condition corresponds also to the maximum of the kinetic energy production in the buffer layer, as shown in Fig. 8. In this figure, the variations of the rate of turbulent kinetic energy production due to the mean flow are shown in LL and BL. The shape of the diagrams follows the classical data (Tennekes & Lumley (1972), Hinze (1975) and references therein), and the values of $\langle uv \rangle^+$, $\langle uv \rangle^+ dU^+/dy^+$ and of the corresponding y^+ , are in excellent agreement with the data of Tennekes & Lumley (1972) and Hinze (1975).

It is worth to remark that, although the monotonic trend of u and $\langle uv \rangle^+$ can contrast some experiments which can give non-monotonic variations of these variables, the values of such quantities and the corresponding y^+ , are comparable with those of the several experiments (Fernholz & Finley (1996)). This discrepancy can be due the fact that, here, the effects of

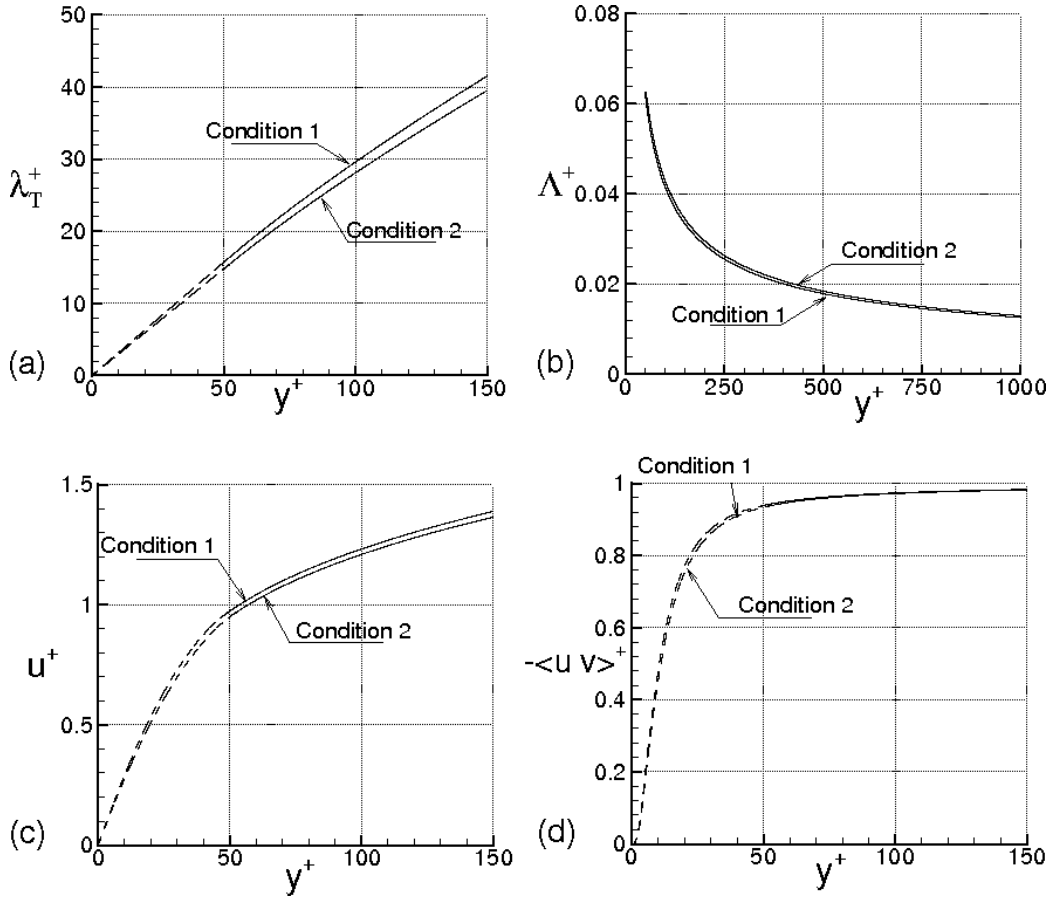


Figure 6: Distribution of the dimensionless variables in the boundary layer: SL dashed lines, TR continuous lines. (a) Taylor scale. (b) Maximal Lyapunov exponent. (c) r.m.s. of fluctuating velocity. (d) Reynolds stress.

non-homogeneity of $\langle u_y(p/\rho + u_j u_j/2) \rangle$ on the kinetic energy equation are neglected in TR, and that the average velocity is analyzed only in the logarithmic range.

8. Conclusions

This work analyzes the turbulent wall laws through the Lyapunov theory of finite scale. The results, valid for fully developed flow with moderate pressure gradient, are subjected to the hypothesis that in the turbulent region,

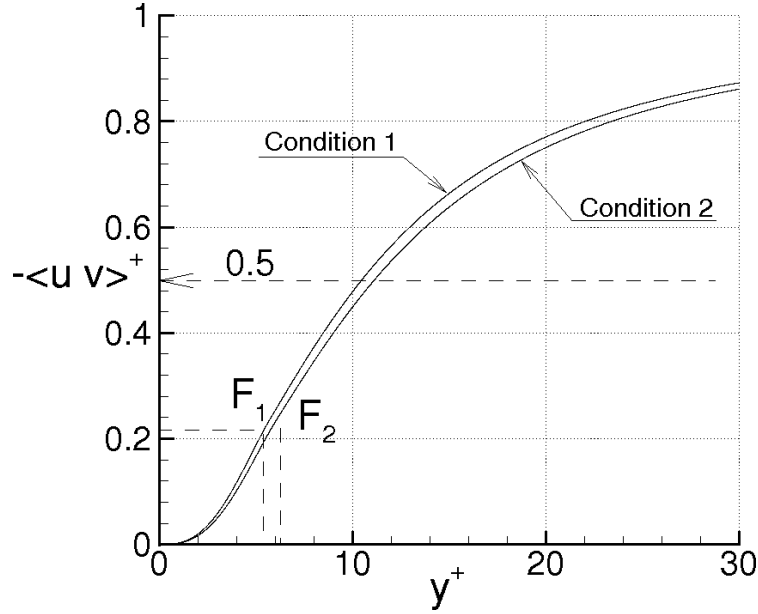


Figure 7: Dimensionless Reynolds stress in the laminar sub-layer and in the buffer region

the energy production due to the average flow is balanced only by the dissipation rate, whereas the non-homogeneity of $\langle u_y(p/\rho + u_j u_j/2) \rangle$ is neglected.

The free parameters of the velocity law, here theoretically calculated through the statistical properties of the velocity correlation functions, and the wall laws, are in very good agreement with the literature. In particular:

- The von Kármán constant, theoretically identified as $k = (dR_T/dy^+)_e \approx 0.4 \div 0.42$, depends on the scale of the spanwise velocity correlation and does not requires the assumption of the logarithmic velocity profile.
- The average velocity law and the distributions of the other dimensionless quantities such as kinetic energy and Reynolds stress in the boundary layer, agree -or at least are comparable- with experiments and direct simulations.

These results, which represent a further application of the analysis presented in de Divitiis (IJES 2011), show that the finite scale Lyapunov analysis can be an adequate theory to explain the wall turbulence.

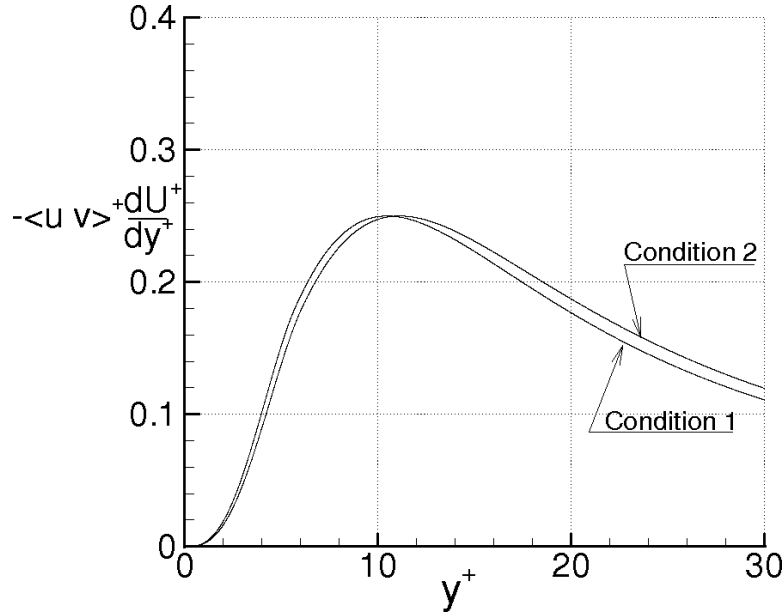


Figure 8: Dimensionless rate of kinetic energy production in the laminar sub-layer and in the buffer region

References

- VON KÁRMÁN, T., Mechanical similitude and turbulence, NACA-TM-611, 1931 (1930).
- NIKURADSE, J., Laws of Turbulent Flow in Smooth Pipes, NASA TT F-10, 359, 1966 (1932).
- REICHARDT, H., Heat Transfer Through Turbulent Friction Layers, NACA-TM-1047, 1943 (1940).
- KLEBANOFF, P. S., DIEHL, Z. W., Some features of artificially thickened fully developed turbulent boundary layers with zero pressure gradient, NACA-TR-1110, 1952.
- KLEBANOFF, P. S., Characteristics of turbulence in boundary layer with zero pressure gradient, NACA-TR-1247, 1955.
- SMITH, D. W., WALKER, J. H., Skin-friction measurements in incompressible flow, NACA-TN-4231, 1958.

- SPALART, P. R., Direct simulation of a turbulent boundary layer up to $Re_\theta = 1410$, *Journal of Fluid Mechanics*, **187**, pp. 61-98, 1988
- FERNHOLZ, H. H., FINLEYT, P. J., The incompressible zero-pressure-gradient turbulent boundary layer: An assessment of the data, *Progress in Aerospace Sciences*, **32**, pp. 245–311, 1996 DOI: 10.1016/0376-0421(95)00007-0
- LANDAU, L. D., LIFSHITZ, M., *Fluid Mechanics*. Pergamon London, England, 1959.
- BARENBLATT, G. I., Scaling laws for fully developed turbulent shear flows. Part 1. Basic hypotheses and analysis., *Journal of Fluid Mechanics*, **248**, pp. 513–520, 1993 DOI: 10.1017/S0022112093000874
- ZAGAROLA, M. V., SMITS, A. J., Mean-flow scaling of turbulent pipe flow., *Journal of Fluid Mechanics*, **373**, pp. 33–79, 1998
- DE DIVITIIS, N., Lyapunov Analysis for Fully Developed Homogeneous Isotropic Turbulence, *Theoretical and Computational Fluid Dynamics*, DOI: 10.1007/s00162-010-0211-9.
- DE DIVITIIS, N., Self-Similarity in Fully Developed Homogeneous Isotropic Turbulence Using the Lyapunov Analysis, *Theoretical and Computational Fluid Dynamics*, DOI: 10.1007/s00162-010-0213-7.
- DE DIVITIIS, N., Steady Homogeneous Turbulence in the Presence of an Average Velocity Gradient, *International Journal of Engineering Science*, DOI: 10.1016/j.ijengsci.2011.10.013
- VON KÁRMÁN, T. & HOWARTH, L., On the Statistical Theory of Isotropic Turbulence., *Proc. Roy. Soc. A*, **164**, 14, 192, 1938.
- BATCHELOR, G. K., *The Theory of Homogeneous Turbulence*. Cambridge University Press, Cambridge, 1953.
- OBERLACK, M., Non-isotropic dissipation in non-homogeneous turbulence, *Journal of Fluid Mechanics* **350**, pp. 351–374, 1997.
- SCHLICHTING, H., GERSTEN, K., *Boundary-Layer Theory*. 8th ed. Springer-Verlag 2004.

- TENNEKES, H., LUMLEY, J. L., *A first course in turbulence*. MIT Press, 1972
- HINZE, J. O., *Turbulence*. McGraw-Hill, 1975
- LEE, M. J., KIM, J., AND MOIN, P., Structure of turbulence at high shear rate, *J. Fluid Mech.*, **216**, pp. 561–583, 1990.
- VENTSEL, E. S., *Theorie des probabilites*. Ed. Mir, CCCP, Moskow, 1973.
- SPALDING, D. B., A Single Formula for the Law of the Wall, *J. Applied Transaction of the ASME*, Series E **83**, pp. 455, 1961
- MUSKER, A. J., Explicit Expression for the smooth wall velocity distribution in a turbulent boundary layer, *AIAA Journal*, **17**, pp. 665-657, 1979

Table 1: Velocity law parameters and comparison of the results. P.R. as for "Present Result".

Parameter	P. R. Condition 1 Eq. (65)	P. R. Condition 2 Eq. (66)	Nikuradse	Zagarola	Barenblatt	Smith	Fernholz
k	0.4233	0.3984	0.4	0.41	0.425	0.461	0.4
C	6.2445	6.4822	5.5	5.2	6.79	7.13	5.1
A	5.2384	5.5652					
B	-4.1226	-4.7166					
$\frac{r_z^*}{\lambda_{Te}}$	3.18233	3.38085					
$\left(\frac{d\lambda_T^+}{dy^+}\right)_e$	0.31423	0.29578					
$\left(\frac{dl_p^+}{dy^+}\right)_e$	0.4235	0.3986					
y_e^+	49.3005	52.3760					
u_e^+	0.96825	0.96825					
$-\langle uv \rangle_e^+$	0.93750	0.93750					
U_e^+	14.8250	15.7498					